ZU CHONGZHI (429 – 500)

by HEINZ KLAUS STRICK, Germany

A special achievement of the Chinese mathematician ZU CHONGZHI was the determination of the circle number π with an accuracy of seven decimal places. This accuracy was only surpassed in the 15th century, almost 1000 years later, by the last great mathematician of the Islamic Middle Ages, AL KASHI, and in Europe at the end of the 16th century by LUDOLPH VAN CEULEN. From 1670 onwards, with the development of differential calculus by NEWTON and LEIBNIZ, completely different calculation methods became available.

ZU CHONGZHI, like his grandfather and father, worked as a civil servant at the Chinese court. They passed on their astronomical knowledge and mathematical skills and knowledge to him. In ancient China, it was believed that the emperor's right to rule came to him from heaven – a proof of the heavenly commission was when a ruler introduced a new calendar.

In his capacity as a high government official, ZU CHONGZHI attempted to develop a calendar that was more in keeping with the solar and lunar cycle than the one previously used. The calendar in force at that time has a 19-year cycle of 235 months (the months had 29 or 30 days; a Chinese month was the time from new moon to new moon) – 12 years with 12 months and 7 years with a thirteenth months. On the basis of his precise astronomical observations, ZU CHONGZHI concluded that a calendar with a cycle of 391 years with a total of 4836 months, including 144 years with 13 months, was better suited to "celestial" conditions. In the scheme he proposed, the average length of the year would have been subject to an error of only 50 seconds compared with the true length of a tropical year.

It can be assumed that ZU CHONGZHI, by measurements for the length of one year, had obtained $365\frac{9\,589}{39\,491}$ days and for the lunar-month $\frac{116\,321}{3\,939}$ days. A year therefore consisted of $12\frac{1\,691\,772\,624}{4\,593\,632\,611}$ months. The fractional part could be shortened to obtain $12\frac{144}{391}$. This means that in 144 out of 391 years an additional lunar month was required.

Despite all the resistance and intrigues at court, ZU CHONGZHI succeeded in convincing his ruler to introduce this complicated calendar cycle. However, since the emperor died in 464 before the change could be implemented and the subsequent ruler did not agree with his predecessor, the new calendar was not introduced.

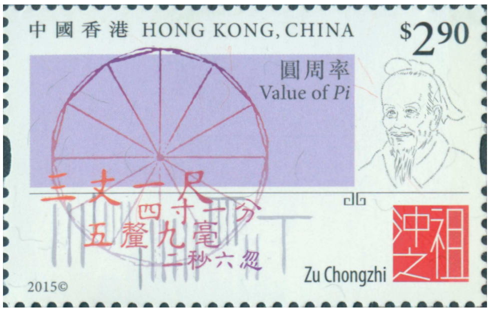
ZU CHONGZHI retired from the imperial court and devoted himself exclusively to mathematics and astronomy. Together with his son ZU GENG, he wrote a mathematics book entitled *Zhui shu* (Method of Interpolation), which was highly acclaimed and was considered one of the famous Ten Classics of Chinese mathematics. Because of its high standards, however, it was soon removed from the compulsory canon of the *Imperial Academy* (anyone wishing to become a civil servant at the Imperial Court had to pass a demanding examination in mathematics). It was reprinted again in 1084, but all traces of the book were lost in the 12th century.

In his book ZU CHONGZHI gave the approximation $\frac{355}{113}$ for the circle number π.

If you write this number as a continued fraction, you get: $\frac{355}{113} = 3 + \frac{16}{113} = 3 + \frac{1}{7 + \frac{1}{16}}$

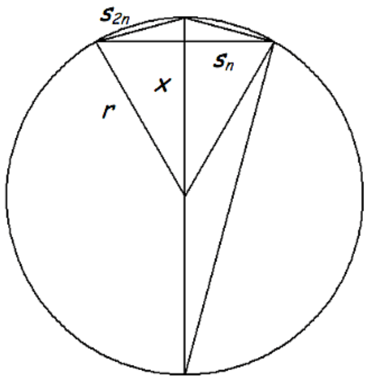
If you omit the last summand in this continued fraction, the result is $3\frac{1}{7} = \frac{22}{7}$ which is an approximation for π already given by ARCHIMEDES.

A source from the 7th century reported that if you looked at a circle with a diameter of 10 000 000 chang, you knew that by the calculations of ZU CHONGZHI the circumference of this circle was more than 31 415 926 chang and less than 31 415 927 chang (1 chang ≈ 3.58 metres).



In his calculations of π ZU CHONGZHI started from the regular hexagon, whose circumference was three times the diameter (the length of the longer diagonal) and then the number of vertices was doubled over and over again.

There is following relationship between the side lengths s_n of a regular n-gon and s_{2n} of a regular $2n$-gon, where x is the distance of the centre from the side of the n-gon:



$s_{2n}{}^2 = 2r \cdot (r - x)$ – from EUCLID's theorem,

$r^2 = \frac{s_n{}^2}{4} + x^2$ – from PYTHAGORAS's theorem,

so $x = \sqrt{r^2 - \frac{s_n{}^2}{4}}$ and therefore $s_{2n} = \sqrt{2r^2 - r \cdot \sqrt{4r^2 - s_n{}^2}}$

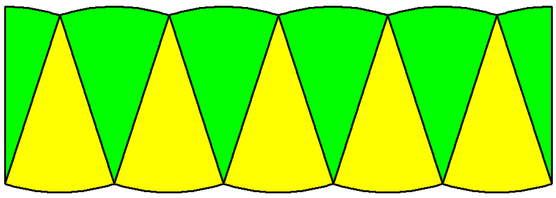
With $r = 1$ one then successively obtains:

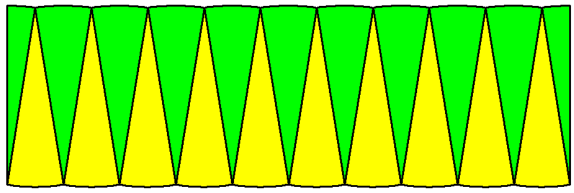
$s_{12} = \sqrt{2 - \sqrt{4 - 1^2}} = \sqrt{2 - \sqrt{3}}$, $s_{24} = \sqrt{2 - \sqrt{4 - (\sqrt{2 - \sqrt{3}})^2}} = \sqrt{2 - \sqrt{2 + \sqrt{3}}}$, ...

n	Side length s_n	$\frac{1}{2} \cdot n \cdot s_n$
6	1	3
12	$\sqrt{2-\sqrt{3}} \approx 0{,}517638...$	3,1058285...
24	$\sqrt{2-\sqrt{2+\sqrt{3}}} \approx 0{,}2610523...$	3,13262861...
48	$\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{3}}}} \approx 0{,}13080625...$	3,13935020...
96	$\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}} \approx 0{,}0654381...$	3,14103195...
192	$\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}} \approx 0{,}03272346...$	3,141452472...

ARCHIMEDES had already proved that a circle has the same area as a rectangle with the sides "half the circle diameter" and "half the circle circumference". If you define the circle number π as the ratio of the circumference of a circle to its diameter, then π approximately equal to half the size of a regular n-gon in the unit circle.

In order to achieve the accuracy of 7 decimal places, ZU CHONGZHI must have worked out – without the tools available to us today – the side length of a regular 24 576-gon - an incredible amount of calculation from today's point of view!





One of the special achievements of ZU CHONGZHI and his son ZU GENG was the derivation of a formula for the volume for the sphere.

Over 200 years earlier LIU HUI (220-280) had said:

- *If you double the volume of this body and take its cube root, you get the diameter of the sphere.*

(This is equivalent to taking $\pi = 3$)

ZU CHONGZHI and ZU GENG gave the formula for the sphere volume as $V = \frac{11}{21} \cdot d^3$ (where $\pi = \frac{22}{7}$).

For the derivation they use the principle:

- *The volumes of two bodies of the same height are in a fixed numerical ratio if the sizes of the intersections of both bodies at the same height are in this numerical ratio.*

This was a generalisation of a principle which was formulated in Europe only 1000 years later by BONAVENTURA CAVALIERI (1598-1647).

In concrete terms, they first divided a cube into eight smaller cubes of the same size. The smaller cubes, in turn, are cut into four smaller pieces by means of several cylindrical cuts, which they compare with parts of a sphere according to the principle given above, and thus determine their volume. It seems particularly significant that ZU CHONGZHI and ZU GENG had recognised the connection between the determination of the area of a circle and the volume of a sphere.

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