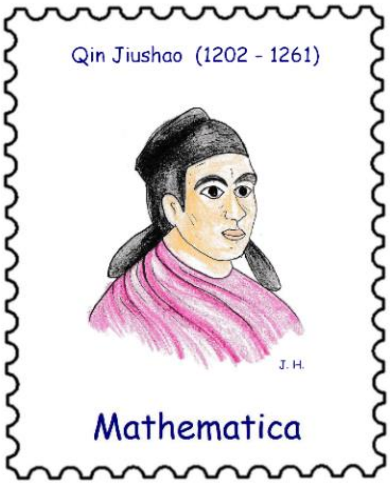
Qin Jiushao (1202 – 1261)

by Heinz Klaus Strick, Germany



Qin Jiushao was born in 1202 as the son of a senior official in Sichuan Province in southwest China. At the age of 17, he volunteered for the army and was involved in a mission that put down an uprising.

After his father's transfer to Hangzhou, the capital of the Song dynasty empire in western China, the son took up opportunities to study mathematics and astronomy.

In 1226, Qin Jiushao returned home with his father, where he could continue his studies.

Anyone wishing to enter the civil service had to pass examinations in six disciplines, one of which was mathematics. The basis for the seven-year study of the applied content was the so-called *Jiuzhang suanshu* (*Nine Chapters of Mathematical Art*), which were compiled and supplemented from the 2nd century BC onwards, among others by Liu Hui (220-280) and Zu Chongzhi (429-501).

Unlike his fellow students, who learnt the methods taught to them by heart in order to pass the exams, Qin Jiushao saw through the mathematical methods. His talent was not only shown in mathematics, but equally in poetry, music and architecture. And at the same time, the unrestrained and unpredictable young man was considered an excellent horseman, swordsman and archer. When the Mongol armies of Genghis Khan threatened the Sichuan province, he became commander of a unit to defend the country.

Qin Jiushao was entrusted with administrative tasks in various provinces, which he performed brutally, so that there was even an uprising against him. The fact that he gained great wealth through the illegal sale of salt stocks did not detract from his career in the civil service. Shortly after he took up an important post at the imperial court in the capital Nanjing in 1244, news reached him of his mother's death.

During the traditional three-year mourning period, Qin Jiushao wrote the masterpiece *Shushu jiuzhang* (*Mathematical Treatise in Nine Chapters*), which appeared in 1247.

It was certainly no coincidence that there were exactly nine chapters (as well as nine selected tasks in each chapter), as this corresponded to the work of his great role model Liu Hui, except for the order of the topics.

In 1254, Qin Jiushao returned to government service for a few months. His assignment as governor in a southern province in 1259 was terminated after 100 days for abuse of office, though the new fortune acquired through bribery secured him further financially carefree years.

In the following year, he succeeded in being entrusted with an administrative post again – as an employee of his friend WU QIAN, who was a naval officer. After WU QIAN's dismissal from the civil service, QIN JIUSHAO was transferred to Meixian in the southern Chinese province of Guangtong, where he died in 1261.

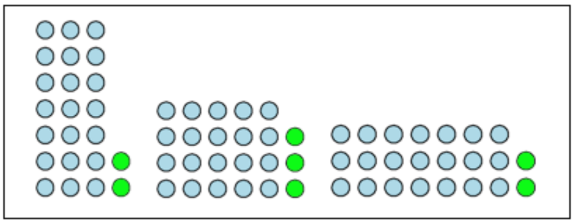
Letterpress printing was invented in China in the 7th century and further developed into printing with movable type in the 11th century.

Why QIN JIUSHAO's work – unlike many other books of the time – was not printed can no longer be clarified. The book was often reproduced and spread to Japan and Korea.

The version we have today was published in 1842 – reconstructed from a Korean copy.

In the *first chapter* of the *Shushu jiuzhang*, QIN JIUSHAO dealt with a class of problems first described by the Chinese mathematician SUN ZI (400-460):

> *An unknown number of objects is given.*
> *Collecting them in groups of* 3 *leaves* 2*, in groups of* 5 *leaves* 3 *and in groups of* 7 *leaves* 2*.*
>
> *How many objects are there?*



Written down in modern notation: We are looking for the smallest natural number *n* with $n \equiv 2 \pmod 3 \wedge n \equiv 3 \pmod 5 \wedge n \equiv 2 \pmod 7$.

SUN ZI gave the following procedure for determining the number sought:

> *Multiply the number of objects left in the* 3*-grouping by* 70*, add to this the product of the number of objects left in the* 5*-grouping by* 21 *and then add to this the product of the number of objects left in the* 7*-grouping by* 15*.*

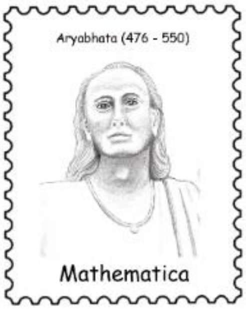
The smallest possible number of objects is obtained by subtracting as large a multiple as possible of $105 = 3 \cdot 5 \cdot 7$ from the result.

Here then: $n = 2 \cdot 70 + 3 \cdot 21 + 2 \cdot 15 - 2 \cdot 105 = 23$.

The factors 70, 21, 105 result from the following conditions:

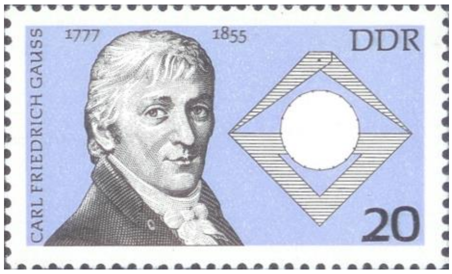
$70 \equiv 1 \pmod 3 \wedge 70 \equiv 0 \pmod 5 \wedge 70 \equiv 0 \pmod 7$, $21 \equiv 1 \pmod 5 \wedge 21 \equiv 0 \pmod 3 \wedge 21 \equiv 0 \pmod 7$, $15 \equiv 1 \pmod 7 \wedge 15 \equiv 0 \pmod 3 \wedge 15 \equiv 0 \pmod 5$.

This procedure described here in the example is generally applicable if the size of the groups under consideration are pairwise coprime. Tasks of this type can also be found 100 years later in ARYABHATA and 750 years later in LEONARDO OF PISA (FIBONACCI).



QIN JIUSHAO is considered to be the first mathematician to solve problems even in the case that the sizes are not pairwise coprime. The method he developed is called the *Chinese Remainder Theorem* in the technical literature.

Five hundred years later, QIN JIUSHAO's method was rediscovered by LEONARD EULER and conclusively treated by CARL FRIEDRICH GAUSS in his *Disquisitiones* in 1801.



One of the tasks of QIN JIUSHAO was as follows:

> *Three farmers (A, B, C) harvest the same amount of rice from their fields. They offer this for sale at different places; different units of volume apply there in each case.*
>
> *A sells his rice at the official market of his own prefecture, where it is measured in units of* 1 *hu* = 83 *sheng; he is left with* 32 *sheng.*
>
> *B offers his rice to the villagers of Anji, where it is measured in units of* 1 *hu* = 110 *sheng. He still has* 7 tou (= 70 *sheng*) *left.*
>
> *C sells his rice to a middleman from Pingjiang, where one unit equals* 1 *hu* = 135 *sheng; he still has* 3 *tou* (= 30 *sheng*) *left.*
>
> *How much rice did each farmer originally have and how much did each sell?*

Solution: Each of the three farmers had harvested 2460 tou (= 24600 sheng) of rice.

A was able to sell 296 hu of 83 sheng each, i.e. 24568 sheng, and had 32 sheng left.

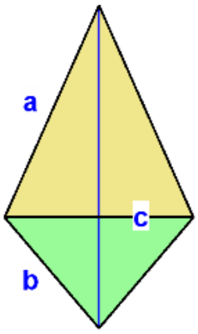
B sold 223 hu of 110 sheng each, i.e. 24530 sheng, and had 70 sheng left.

C sold 182 hu of 135 sheng each, i.e. 24570 sheng, and had 30 sheng left over.

The *second chapter* of the *Shushu jiuzhang* contained tasks on "celestial phenomena", i.e. problems that have to do with astronomical calculations such as determining the length of shadows at certain places or the phases of visibility of Jupiter, but also with rainfall measurements.

In the *third chapter*, QIN JIUSHAO dealt with area determinations. For the solution of the first task, the area determination of a kite quadrilateral, he gave a remarkable relationship:

> *The area of the figure* $X = \sqrt{A} + \sqrt{B}$ *on the right is given by an equation of degree 4 :* $-X^4 + 2 \cdot (A+B) \cdot X^2 - (A-B)^2 = 0$*, where by* $A = (a^2 - (\frac{c}{2})^2) \cdot (\frac{c}{2})^2$ *and* $B = (b^2 - (\frac{c}{2})^2) \cdot (\frac{c}{2})^2$ *the squares of the areas of the two parts are calculated.*



Incidentally, this biquadratic equation also applies to the area X of a circular ring, where A, B denote the squares of the areas of the outer and inner circles, respectively.

For the area S of a triangle with sides a, b, c, QIN JIUSHAO gave the formula

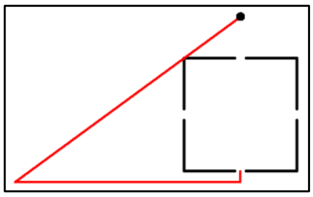
$$S = \sqrt{\frac{1}{4} \cdot [c^2 \cdot a^2 - (\frac{c^2 + a^2 - b^2}{2})^2]}.$$

This is nothing other than HERON's formula. He also developed a formula with which one can determine the area of a quadrilateral from the lengths of the four sides and one of the heights.

In the *fourth chapter*, QIN JIUSHAO dealt with the determination of distances from inaccessible points. In this context, he considered a task that was probably inspired by the following problem from *Jiuzhang suanshu*.

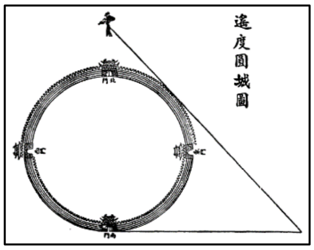
It states:

> *In a city with a square ground plan, there is a tree at a distance of* 20 *bu from the north gate. If you walk* 14 *bu south from the southern city gate and then west at* 1775 *bu, you will see the tree behind the northwest corner of the city wall.*



QIN JIUSHAO's problem is:

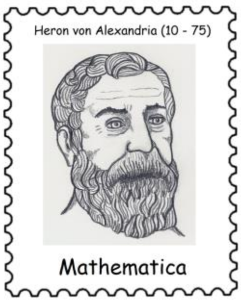
> *A circular walled city of unknown diameter has gates in the four cardinal directions. Three Li north of the north gate is a tree. If you turn around and walk nine Li east immediately after leaving the southern gate, the tree will just come into view. Determine the circumference and diameter of the city wall.*



Without further comment or reference, QIN JIUSHAO stated that the problem comes down to the equation $x^{10} + 15x^8 + 72x^6 - 864x^4 - 11664x^2 - 34992 = 0$, i.e. a 10th degree equation, where he defines the diameter to be x^2. The solution is: $x^2 = 9$.

Chapters 5, 6, 7 and 8 contained problems similar to those already contained in *Jiuzhang suanshu*, including the topics

- *Taxes*: Calculating the tax burden;
- *Money*: Determining the exchange rate of paper money (paper money was invented in China, first issued in 1024 when coinage became scarce), cost of transporting grain;
- *Fortress construction and buildings*: cost of materials for building a fortress, excavation for building a canal, building a dyke;
- *Military*: Planning army camps and battle formations.

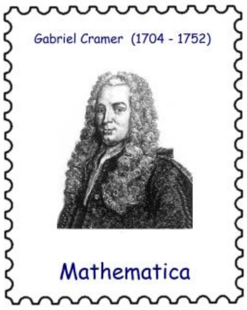


At the end of *chapter eight* is the following problem:

> *A certain number of cotton bales are stored in a warehouse, from which a certain number of army uniforms are to be made. If one takes 8 bales each to make clothes for 6 soldiers, then 160 bales are missing. If you take 9 bales each to make clothes for 7 soldiers, then 560 bales remain. The number of cotton bales and the number of soldiers to be clothed are needed.*

If we use the variable *x* to denote the number of bales and *y* to denote the number of soldiers, we obtain the two equations $x = \frac{y}{6} \cdot 8 - 160 \wedge x = \frac{y}{7} \cdot 9 + 560$. These lead to the solution $x = 20000$ and $y = 15120$.

QIN JIUSHAO solved the problem in a general form reminiscent of CRAMER's rule (named after the Swiss mathematician GABRIEL CRAMER, 1704-1752). Such a general approach was only published again afterwards by the Japanese mathematician SEKI KOWA (1642-1708), who developed the determinant method ten years before GOTTFRIED WILHELM LEIBNIZ.



In the *ninth chapter*, problems related to the trade of commodities were examined, leading to systems of linear equations.

> *A trader enters into three transactions, each involving an amount of 1,470,000 kuan.*
> *In the first he buys 3500 bundles of wool, 2200 chin (pounds) of tortoise shell and 375 boxes of incense, in the second 2970 bundles of wool, 2130 chin of tortoise shell and 3056 ¼ boxes of incense, in the third 3200 bundles of wool, 1500 chin of tortoise shell and 3750 boxes of incense.*
> *What is the value of one bundle of wool, one chin tortoise shell and one box of incense respectively?*

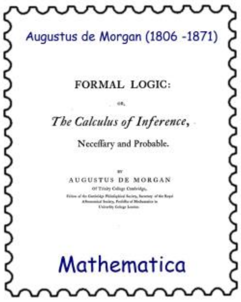
The problem can be represented as a linear system of equations with three equations and three variables. QIN JIUSHAO wrote down the coefficients in tabular form (see below, middle) and solved the system by elementary column transformations – now called the GAUSS elimination method. (Solution: $x = 300$, $y = 180$, $z = 64$)

The calculations with the coefficients were carried out with the help of the abacus, the so-called *suanpan*, which was common in China.

It is remarkable that QIN JIUSHAO was the first to note a small circle as a symbol for zero in the tables of the intermediate steps, whereas until then such places were simply left blank.

$$\begin{aligned} 3500x + 2200y + 375z &= 1470000 \\ 2970x + 2130y + 3056\tfrac{1}{4}z &= 1470000 \\ 3200x + 1500y + 3750z &= 1470000 \end{aligned}$$

1470000	1470000	1470000
3200	2970	3500
1500	2130	2200
3750	$3056\frac{1}{4}$	375



The method used for this was usually referred to in our cultural area as the *HORNER scheme* after the suggestion of AUGUSTUS DE MORGAN – named after the English mathematician WILLIAM GEORGE HORNER, who presented this method to the *Royal Society* in 1819 (ten years earlier, the Italian mathematician PAOLO RUFFINI had already published it).

Example: By trial and error, one finds that a solution to the equation $1x^3 + 3x^2 - 5x - 12 = 0$ is between 2 and 3. Therefore, it is obvious to use the approach $x = 2 + y$ to give

$1 \cdot (2+y)^3 + 3 \cdot (2+y)^2 - 5 \cdot (2+y) - 12 = 0$.

The term on the left can be calculated with the help of binomial formulae and then we get $1y^3 + 9y^2 + 19y - 2 = 0$. This equation has a solution that lies in the interval [1 , 2].

If you want to avoid decimals, you can replace y with $10y'$, i.e. the equation $1000y'^3 + 900y'^2 + 190y' - 2 = 0$ and finds that it has a solution that lies in the interval [1 , 2], i.e., the initial equation has a solution in the interval [2.1 , 2.2], etc.

However, the initial equation can also be expressed in the form $(1 \cdot (x+3) \cdot x - 5) \cdot x - 12 = 0$.

To determine values of the term on the left, no powers have to be formed, but only sums and products alternately. Replacing the variable x with $y + 2$ is less costly:

$(1 \cdot ((y+2)+3) \cdot (y+2) - 5) \cdot (y+2) - 12 = 0$.

The transformations can be carried out quickly using the diagram on the right:

$$\begin{aligned} &(1 \cdot ((y+2)+3) \cdot (y+2) - 5) \cdot (y+2) - 12 = 0 \Leftrightarrow \\ &(1 \cdot (y+5) \cdot (y+2) - 5) \cdot (y+2) - 12 = 0 \Leftrightarrow \\ &(1 \cdot (y+5+2) \cdot y + 10 - 5) \cdot (y+2) - 12 = 0 \Leftrightarrow \\ &(1 \cdot (y+7) \cdot y + 5) \cdot (y+2) - 12 = 0 \Leftrightarrow \\ &(1 \cdot (y+7+2) \cdot y + 5 + 14) \cdot y + 10 - 12 = 0 \Leftrightarrow \\ &(1 \cdot (y+9) \cdot y + 19) \cdot y - 2 = 0 \end{aligned}$$

1	3	-5	-12
	2	10	10
1	5	5	- 2
	2	14	
1	7	19	
	2		
1	9		

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